Fourier Analysis

Mar 22, 2022

Review

Def. Let $f \in M(\mathbb{R})$. The Fourier transform of f is

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) \cdot e^{-2\pi i \xi x} \, dx, \quad \xi \in \mathbb{R}$$

Prop 1. Let $f \in M(\mathbb{R})$. Then the following hold:

(1) $f(x+h) \xrightarrow{\mathcal{F}} \hat{f}(\xi) \cdot e^{2\pi i \xi h} \quad \forall\, h \in \mathbb{R}.$

(2) $f(x) \cdot e^{-2\pi i h x} \xrightarrow{\mathcal{F}} \hat{f}(\xi + h), \quad \forall\, h \in \mathbb{R}.$

(3) Let $\delta > 0$. Then

$$f(\delta x) \xrightarrow{\mathcal{F}} \frac{\hat{f}\left(\frac{\xi}{\delta}\right)}{\delta}.$$

(4) Suppose $f' \in M(\mathbb{R})$. Then

$$f'(x) \xrightarrow{\mathcal{F}} \hat{f}(\xi) \cdot (2\pi i \xi)$$

(5) Suppose $x f(x) \in M(\mathbb{R})$. Then

$$f(x) \cdot (-2\pi i x) \xrightarrow{\mathcal{F}} \frac{d \hat{f}(\xi)}{d\xi}$$

Proof: ①: $f(x+h) \xrightarrow{\mathcal{F}} \hat{f}(\xi) \cdot e^{2\pi i \xi h}$, $h \in \mathbb{R}$.

$$\int_{\mathbb{R}} \underbrace{f(x+h)\, e^{-2\pi i \xi x}}_{\substack{\in M(\mathbb{R}) \\ =: g(x)}} dx = \int_{\mathbb{R}} g(x-h)\, dx$$

$$= \int_{\mathbb{R}} f(x)\, e^{-2\pi i \xi (x-h)} dx$$

$$= \int_{\mathbb{R}} f(x)\, e^{-2\pi i \xi x} \cdot e^{2\pi i \xi h} dx$$

$$= \hat{f}(\xi)\, e^{2\pi i \xi h}.$$

④: If $f' \in M(\mathbb{R})$, then $f'(x) \xrightarrow{\mathcal{F}} \hat{f}(\xi) \cdot (2\pi i \xi)$.

$$\int_{-\infty}^{\infty} f'(x)\, e^{-2\pi i \xi x} dx = f(x)\, e^{-2\pi i \xi x} \Big|_{x=-\infty}^{\infty}$$

$$- \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x} \cdot (-2\pi i \xi)\, dx$$

$$= 2\pi i \xi\, \hat{f}(\xi)$$

(5): Suppose $x f(x) \in \mathcal{M}(\mathbb{R})$, then

$$f(x)(-2\pi i x) \xrightarrow{\mathcal{F}} \frac{d \hat{f}(\xi)}{d\xi}.$$

To show this formula, we need a special version of Dominated convergence Thm in real analysis:

Thm: Let $(g_t)_{t \in (a,b)}$ be a family of functions in $\mathcal{M}(\mathbb{R})$. Suppose that

(1) $\lim_{t \to t_0} g_t(x) = g_{t_0}(x)$ for every $x \in \mathbb{R}$

(2) $\exists\ H \in \mathcal{M}(\mathbb{R})$ such that

$$|g_t(x)| \leq H(x) \quad \text{for all } t \in (a,b) \text{ and } x \in \mathbb{R}.$$

Then $\lim_{t \to t_0} \int_{-\infty}^{\infty} g_t(x)\, dx = \int_{-\infty}^{\infty} g_{t_0}(x)\, dx.$

Now we prove the formula in (5).

$$\frac{\hat{f}(\xi+\Delta\xi)-\hat{f}(\xi)}{\Delta\xi}$$

$$= \frac{\int_{\mathbb{R}} f(x) e^{-2\pi i(\xi+\Delta\xi)x}\,dx - \int_{\mathbb{R}} f(x)\, e^{-2\pi i \xi x}\,dx}{\Delta\xi}$$

$$= \int_{\mathbb{R}} f(x) \cdot \frac{e^{-2\pi i(\xi+\Delta\xi)x} - e^{-2\pi i \xi x}}{\Delta\xi}\,dx$$

$$= \int_{\mathbb{R}} \underbrace{f(x) \cdot e^{-2\pi i \xi x}\, \frac{e^{-2\pi i \Delta\xi x} - 1}{\Delta\xi}}_{=:\, g_{\Delta\xi}(x)}\,dx$$

$$=: g_{\Delta\xi}(x).$$

Then

$$\lim_{\Delta\xi \to 0} g_{\Delta\xi}(x) = f(x)\, e^{-2\pi i \xi x} (-2\pi i x).$$

To see that $\left| g_{\Delta\xi}(x) \right|$ is bounded above by a function in $\mathcal{M}(\mathbb{R})$, let us estimate

$$\left| \frac{e^{-2\pi i \Delta\xi x} - 1}{\Delta\xi} \right| = \left| e^{-\pi i \Delta\xi x} \cdot \frac{e^{-\pi i \Delta\xi x} - e^{\pi i \Delta\xi x}}{\Delta\xi} \right|$$

$$= \left| \frac{e^{-\pi i \Delta\xi x} - e^{\pi i \Delta\xi x}}{\Delta\xi} \right|$$

$$= \left| \frac{2 \sin(\pi \Delta\xi x)}{\Delta\xi} \right|$$

$$\leqslant 2\pi |x| \quad \left(\text{since } |\sin y| \leqslant |y| \right)$$

Hence

$$\left| g_{\Delta\xi}(x) \right| \leq |f(x)| \, 2\pi |x|$$

$\left(\text{but } |f(x)| \cdot 2\pi |x| \in \mathcal{M}(\mathbb{R}) \right).$

By the Dominated convergence Thm

$$\lim_{\Delta\xi \to 0} \int_{\mathbb{R}} g_{\Delta\xi}(x) \, dx$$

$$= \int_{\mathbb{R}} \lim_{\Delta\xi \to 0} g_{\Delta\xi}(x) \, dx$$

$$= \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \cdot (-2\pi i x) \, dx$$

That is,

$$\frac{d \hat{f}(\xi)}{d \xi} = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} (-2\pi i x) \, dx.$$

$\boxed{/\!/}$

- Further examples

Example 1: Find the Fourier transform of

$$f(x) = e^{-|x|}.$$

Solution: By definition,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-|x|} e^{-2\pi i \xi x} \, dx$$

$$= \int_0^{\infty} e^{-x} e^{-2\pi i \xi x} \, dx$$

$$+ \int_{-\infty}^{0} e^{x} e^{-2\pi i \xi x} \, dx$$

$$= \int_0^{\infty} e^{-x(1+2\pi i \xi)} \, dx$$

$$+\int_{-\infty}^{0} e^{x(1-2\pi i\xi)}\,dx$$

$$= \left.\frac{e^{-x(1+2\pi i\xi)}}{-(1+2\pi i\xi)}\right|_{0}^{\infty}$$

$$+ \left.\frac{e^{x(1-2\pi i\xi)}}{1-2\pi i\xi}\right|_{-\infty}^{0}$$

$$= \frac{1}{1+2\pi i\xi} + \frac{1}{1-2\pi i\xi}$$

$$= \frac{2}{(1+2\pi i\xi)(1-2\pi i\xi)} = \frac{2}{1+4\pi^2\xi^2}$$

Example 2. Fourier transform of

$$f(x) = e^{-\pi x^2} \quad \text{on } \mathbb{R}.$$

We would like to show

$$\hat{f}(\xi) = e^{-\pi \xi^2}.$$

We first calculate

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i 0 \cdot x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx$$

Notice that

$$\left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-\pi y^2} dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} \, dx\, dy \quad (*)$$

Using polar coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad \begin{pmatrix} 0 \le r < \infty \\ 0 < \theta < 2\pi \end{pmatrix}$$

$$(*) = \int_0^{2\pi} \left(\int_0^{\infty} e^{-\pi r^2} r\, dr \right) d\theta$$

$$= \int_0^{2\pi} \cdot \left(\frac{e^{-\pi r^2}}{-2\pi} \Bigg|_0^{\infty} \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \, d\theta = 1 .$$

Hence $\displaystyle\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$

So $\hat{f}(0) = 1.$

Now notice that $x e^{-\pi x^2} \in \mathcal{M}(\mathbb{R})$, so

$$\frac{d\hat{f}(\xi)}{d\xi} = \int_{-\infty}^{\infty} (-2\pi i x)\, e^{-\pi x^2} e^{-2\pi i \xi x}\, dx$$

(using (5) of Prop 1)

$$= \int_{-\infty}^{\infty} i \, \underbrace{\left(e^{-\pi x^2}\right)'}_{= f(x)'} \cdot e^{-2\pi i \xi x}\, dx$$

$$= i\, \hat{f}(\xi) \cdot (2\pi i \xi)$$

$$= -2\pi \xi\, \hat{f}(\xi)$$

Hence $\hat{f}(\xi)$ satisfies the following ODE:

$$\frac{d\,\hat{f}(\xi)}{d\xi} = -2\pi\xi\,\hat{f}(\xi)$$

So $$\frac{d\,\hat{f}(\xi)}{d\xi}\cdot\frac{1}{\hat{f}(\xi)} = -2\pi\xi.$$

thus

$$\frac{d\ \ln\hat{f}(\xi)}{d\xi} = -2\pi\xi$$

Hence $$\ln\hat{f}(\xi) = -\pi\xi^2 + C$$

$$\Rightarrow \quad \hat{f}(\xi) = \tilde{C}\cdot e^{-\pi\xi^2}.$$

but $\hat{f}(0) = 1$

So $\hat{f}(\xi) = e^{-\pi\xi^2}.$

$\blacksquare$

§ 5.3 Inversion formula.

Thm (Fourier inversion formula)

Let $f \in \mathcal{M}(\mathbb{R})$. Suppose that $\hat{f} \in \mathcal{M}(\mathbb{R})$.

Then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi, \quad \forall\, x \in \mathbb{R}.$$

To prove the above theorem, let us introduce one concept.

Def. Let $f, g \in \mathcal{M}(\mathbb{R})$. Set

$$f * g(x) := \int_{\mathbb{R}} f(x-y)\, g(y)\, dy.$$

<u>Prop 2</u> : Let $f, g \in \mathcal{M}(\mathbb{R})$. Then

(1) $f * g = g * f$.

(2) $f * g \in \mathcal{M}(\mathbb{R})$.

(3) $\widehat{f * g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$.